Solution 6

Supplementary Problems

1. The rotation by an angle θ in anticlockwise direction is given by $(x, y) = (\cos\theta\ u - \sin\theta\ v, \sin\theta\ u + \cos\theta\ v)$. Verify that rotation leaves the area unchanged.

   Solution. Let G be a region in the plane. The area of G is defined to be $\iint_G 1\, dA(u, v)$. After the rotation G to D, and the area of D is $\iint_D 1\, dA(x, y)$. The Jacobian of the change of variables $\dfrac{\partial(x, y)}{\partial(u, v)}$ is easily calculated to be 1. Therefore,

$$|D| = \iint_D 1\, dA(x, y) = \iint_G 1 \times 1\, dA(u, v) = |G| \ .$$

   Note. It is easy to verify that other Euclidean motions such as translations and reflections also leave the area unchanged. Their Jacobians are all equal to 1.

2. Let D be the region bounded by four lines $y = ax + b_1, y = ax + b_2, y = cx + d_1, y = cx + d_2$ where you may assume $c > a > 0, b_1 < b_2$ and $d_1 < d_2$. Show the area of D is given by $(b_2 - b_1)(d_2 - d_1)/(c - a)$.

   Solution. Letting $u = y - ax$ and $v = y - cx$, G is the rectangle $[b_1, b_2] \times [d_1, d_2]$. We have

$$\frac{\partial(u, v)}{\partial(x, y)} = (-a) \times 1 - (-c) \times 1 = c - a \ .$$

   By the change of variables formula, the area of D is

$$\iint_G 1 \times \frac{1}{c - a}\, dA(u, v) = \int_{b_1}^{b_2} \int_{d_1}^{d_2} 1 \times \frac{1}{c - a}\, dv du = \frac{(b_2 - b_1)(d_2 - d_1)}{c - a} \ .$$